

Application of Mathematical Inversion Theory in Geophysics

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Abstract: Under complex geological conditions and multi-source observation backgrounds, geophysical inversion problems exhibit strong nonlinearity, non-uniqueness of solutions, and high parameter dimensionality, which impose higher adaptability requirements on mathematical inversion theory. This study systematically explores the mathematical foundations and numerical implementation pathways of geophysical inversion, covering the classification structure of inversion problems, commonly used algorithms, the forward-inversion collaborative mechanism, and regularization strategies. It further analyzes the development trends of multi-physical field integration, data-driven modeling, and dynamic inversion. The study points out that inversion structures integrating physical constraints and intelligent methods significantly enhance the stability and accuracy of solutions, promoting the evolution of geophysical modeling toward multi-scale integration and intelligent optimization.

Keywords: mathematical inversion; geophysical model; multi-source data fusion; regularization; dynamic modeling; numerical optimization

Introduction

Geophysical inversion, as an important approach linking surface observations with subsurface structure reconstruction, has become a core technology in resource exploration, geological evaluation, and environmental monitoring. With the refinement of observation methods and the increasing complexity of model structures, traditional inversion methods have gradually shown limitations in dealing with nonlinear coupling, high-dimensional parameters, and data uncertainties. Mathematical inversion theory provides a solid theoretical foundation and an extensible computational framework to address these challenges, becoming a major driving force for the advancement of current geophysical modeling methods. In practical applications, observation data obtained from different physical fields have distinct advantages in response characteristics and resolution, promoting the evolution of inversion systems from single-source information to multi-source integration. Meanwhile, the introduction of data-driven techniques such as deep learning and Bayesian inference has led to significant breakthroughs in robustness, efficiency, and accuracy of inversion methods. Therefore, systematically exploring the adaptive construction pathways of mathematical inversion theory in geophysics from the perspective of inversion structure optimization and numerical mechanism collaboration not only holds significant theoretical value but also provides practical guidance for modeling in complex geological scenarios.

1. Basic Structure and Adaptability Analysis of Mathematical Inversion Theory

1.1 Definition and Classification of Mathematical Inversion Problems

Mathematical inversion problems constitute the core theoretical foundation for analyzing the “effect-to-cause” process in natural systems, aiming to infer the unknown internal structures or parameters of a system based on observational data. Unlike forward problems, inversion problems are typically characterized by non-uniqueness, high dimensionality, and strong ill-posedness, which prevent the simultaneous fulfillment of solution existence, uniqueness, and stability. In geophysics, inversion problems are widely applied to reconstruct subsurface geological structures, electrical property distributions, and velocity models. Their physical essence lies in estimating the spatial properties of deep media by utilizing limited information measured at the surface, such as seismic

waves, gravity, magnetic fields, and electromagnetic responses, serving as a crucial bridge between physical observation and geological interpretation [1].

According to the types of inversion variables, the relationships between models and data, and the approaches to uncertainty handling, geophysical inversion problems can be further classified into several categories. In the linear and nonlinear classification, linear inversion problems are often solved using matrix operations or singular value decomposition, with a clearly defined algebraic structure in the solution process; whereas nonlinear problems involve function iteration and numerical approximation, requiring optimization algorithms to progressively approach the optimal solution of the objective function. In the deterministic and probabilistic classification, the former pursues a unique optimal solution, while the latter characterizes the uncertainty of the solution space through probability distributions. Moreover, based on different inversion objectives, inversion problems can be divided into parameter inversion and structural inversion: the former focuses on obtaining the quantitative values of physical properties, whereas the latter emphasizes the identification of geometric or topological features of models, providing essential support for modeling complex geological bodies.

1.2 Theoretical Foundations of Common Inversion Algorithms

In solving inversion problems, the choice of algorithm directly determines the stability of solutions, convergence speed, and physical interpretability. Traditional methods such as the least-squares method and gradient descent are widely used in linear inversion problems due to their computational simplicity; however, their sensitivity to initial values and tendency to become trapped in local minima are evident in nonlinear inversion. To address these issues, various iterative algorithms based on optimization theory have been developed, such as the conjugate gradient method, quasi-Newton method, and trust-region method. By introducing more refined step-size adjustment mechanisms and local search strategies, these methods effectively enhance the solvability of nonlinear models. Meanwhile, to mitigate the ill-posedness of inversion problems, regularization techniques have been systematically introduced. By adding prior constraint terms to the objective function, these techniques stabilize the structure of solutions. Typical methods include Tikhonov regularization, total variation regularization, and minimum support regularization.

In recent years, the introduction of probabilistic methods and the Bayesian framework has shifted the focus of inversion from merely seeking the “optimality” of solutions to also addressing their uncertainty structure. Bayesian inversion constructs a posterior probability density function by combining prior probabilities with likelihood functions and then estimates the distribution characteristics of solutions using methods such as Markov Chain Monte Carlo (MCMC) or variational inference, providing a more robust pathway for parameter identification in complex data environments. Furthermore, the integration of machine learning and artificial intelligence technologies into inversion has injected new vitality into algorithmic systems. For instance, end-to-end inversion models based on deep neural networks can learn the nonlinear mapping between data and models during the training phase, significantly reducing the large number of iterative computations required by traditional inversion, thereby greatly improving computational efficiency and inversion accuracy for high-dimensional data [2].

1.3 Adaptability Requirements of Geophysical Models for Inversion Theory

Geophysical models are characterized by typical complexity, manifested in the coexistence of multi-scale structures, strong coupling of physical parameters, and nonlinear responses of observational data, which place higher demands on the adaptability of inversion theory in modeling and solution mechanisms. Particularly when dealing with velocity discontinuities, heterogeneous layered structures, or multi-field coupling effects in real geological environments, traditional inversion theory faces challenges such as unstable solutions and insufficient analytical accuracy. Therefore, modern inversion theory requires the introduction of more physically oriented prior knowledge and constraint mechanisms during the model construction stage. By incorporating physical field-driven approaches, structural coupling, and multi-source data fusion, it achieves a more accurate characterization of geophysical systems. For example, multi-wave and multi-component seismic inversion can integrate information from P-waves, S-waves, and converted waves, enhancing subsurface structural resolution from different perspectives.

To address the computational burden imposed by high-dimensional complex models, current research trends focus on combining rapid modeling with efficient inversion to develop modular

inversion frameworks adaptable to various geological scenarios. On one hand, constructing simplified model spaces (model space reduction) and compressed parameter spaces (dimensionality reduction) significantly reduces inversion dimensionality. On the other hand, physics-informed neural networks (PINNs), as model-driven inversion methods based on deep learning, introduce partial differential equations and boundary conditions to build physics-constrained loss functions, thereby enhancing model generalization while maintaining the physical consistency of solutions. Additionally, when dealing with incomplete data, noise contamination, or observational blind spots, inversion completion strategies based on generative models, such as Variational Autoencoders (VAEs) and Generative Adversarial Networks (GANs), demonstrate strong capabilities in data reconstruction and structural compensation, providing new pathways for extending the adaptability of inversion theory.

2. Numerical Implementation and Structural Optimization in Geophysical Inversion

2.1 Construction of Forward Modeling and Inversion Collaborative Mechanisms

Forward modeling serves as the fundamental prerequisite for geophysical inversion, and its quality directly determines the reliability and physical consistency of inversion results. In numerical inversion workflows, the task of the forward model is to map given physical parameters to observational responses, thereby providing the computational basis for the objective function in the inversion process. Common forward modeling methods in geophysical problems include finite difference methods, finite element methods, spectral methods, and ray tracing methods, each with distinct advantages suitable for different geological environments and physical scenarios. When constructing the forward modeling module, it is necessary to fully consider wavefield propagation, scattering effects, and boundary condition treatments to ensure computational stability and analytical accuracy. Accurate forward simulations not only improve the degree of data fitting but also provide a rigorous physical foundation for subsequent inversion constraints and error control [3].

The essence of the inversion process is the reverse use of the forward model; therefore, constructing an efficient collaborative mechanism between forward modeling and inversion is crucial. Current research trends favor the establishment of coupled solution frameworks that achieve high computational integration of forward and inversion modules through automatic differentiation, adjoint gradient techniques, or joint optimization pathways. Automatic differentiation effectively avoids the errors and inconsistencies introduced by manual gradient derivation, enabling more accurate and efficient computation of objective function gradients. The adjoint method provides scalable sensitivity analysis schemes for large-scale parameter models, significantly reducing computational resource consumption. On this basis, structural optimization frameworks model error propagation pathways, enhancing solution robustness and the physical interpretability of model fitting, thus achieving an integrated fusion of forward modeling, inversion, and optimization.

2.2 Design of Regularization Strategies in High-Dimensional Parameter Spaces

The issue of high-dimensional parameter spaces in geophysical inversion arises from the need for detailed characterization of complex subsurface structures. While such high-degree-of-freedom modeling enhances representational capability, it also leads to severe ill-posedness and solution instability. As a core mechanism for handling ill-posed problems, regularization techniques must be designed specifically based on data characteristics and geological prior information. In cases with high parameter dimensionality, classical Tikhonov regularization fails to effectively suppress model oscillations and overfitting. Structural regularization, by introducing prior geometric information or model sparsity, optimizes the distribution structure of the solution space. For example, total variation regularization helps preserve sharp interfaces without excessive smoothing, while minimum support regularization is suitable for high-resolution reconstruction of localized anomalous bodies.

To address the challenge of expressing prior knowledge in high-dimensional spaces, Bayesian regularization methods provide a more flexible modeling approach. By constructing probability density functions to constrain the solution space, they simultaneously account for both the most likely solution and the expression of uncertainty structures. Within this framework, the regularization term is no longer a fixed penalty function but a dynamic parameter field reflecting the relationship between physical models and data. With the advancement of deep learning and data-driven technologies, regularization strategies are gradually evolving toward a learnable paradigm, where training data are used to automatically extract prior distributions and model features, demonstrating stronger generalization

capability in high-dimensional scenarios. Multi-scale regularization methods are also increasingly applied to complex geological structures. By constructing penalty functions within scale-decomposed domains, these methods achieve coordinated regulation of local structures, effectively enhancing the unified modeling capability for both detailed features and global trends.

2.3 Stability and Convergence Analysis of Iterative Solution Algorithms

Commonly used iterative algorithms in geophysical inversion, such as the Gauss-Newton method, quasi-Newton method, and conjugate gradient method, have their stability and convergence significantly affecting the physical rationality and numerical efficiency of inversion results. Due to the nonlinearity, non-convexity, and sensitivity to data perturbations inherent in inversion problems, iterative paths are prone to deviating from the global optimum or becoming trapped in local minima, which imposes higher requirements on the stability mechanism design of solution algorithms. Stability is typically evaluated through spectral analysis of the Jacobian and Hessian matrices, essentially aiming to control oscillations and divergence risks during parameter updates. In large-scale problems, the introduction of step-size control, search direction constraints, and preconditioned transformations can effectively improve numerical behavior and enhance algorithm robustness ^[4].

Strategies for improving convergence rely on refined characterization of the objective function surface and efficient utilization of gradient information. Multi-scale line search strategies accelerate global convergence while ensuring descent directions, whereas sensitivity update mechanisms based on adjoint states significantly reduce the computational cost of gradients in high-dimensional models. In inversion environments with high uncertainty and significant data errors, robust optimization methods and variational inference algorithms exhibit superior convergence stability, maintaining stable parameter update paths even in the presence of both model and observational errors. Recent studies have increasingly incorporated meta-learning and adaptive optimization mechanisms into inversion iterative frameworks. By learning optimal solution strategies under different scenarios, these methods enable algorithms to maintain high adaptability and transferability under heterogeneous data conditions, providing theoretical support and practical feasibility for constructing efficient and stable inversion solution systems.

3. Construction and Evolution of Multi-Source Data Fusion Inversion Models

3.1 Coupling Mechanisms of Multi-Physical Field Inversion

Multi-physical field inversion achieves more comprehensive and refined analytical modeling of subsurface medium properties by integrating observational data collected from multiple different physical processes, representing a significant development trend in the current field of geophysical inversion. Due to differences in sensitivity and response patterns of different physical fields to target regions, a single physical field often suffers from issues such as information redundancy, limited resolution, or insensitivity to certain parameters. Multi-physical field fusion enables complementary information integration at the data level, improving the identifiability of model parameters and the stability of inversion results. The construction of coupling mechanisms needs to address key issues such as data dimensionality, response scale, and physical model inconsistency, and achieve unified interpretation of various data through joint objective functions or collaborative optimization strategies.

In numerical implementation, joint inversion models can generally be classified into three typical forms: structural coupling, parameter coupling, and full coupling. Structural coupling emphasizes geometric consistency of models under different physical fields, achieving joint updates through shared model topology constraints. Parameter coupling integrates data from different fields by encoding them jointly based on the mapping relationships among various physical properties, revealing the intrinsic co-evolutionary characteristics of these properties. Full coupling establishes a unified physical description at the equation level, requiring unified modeling and joint discretization of different physical processes, which involves higher computational complexity and theoretical challenges. Coupled inversion methods not only enhance solution accuracy but also significantly improve geological consistency and interpretability, providing an effective approach for addressing high-dimensional nonlinear inversion problems in complex geological environments ^[5].

3.2 Structural Construction of Data-Driven Inversion Models

Data-driven inversion methods, based on big data and machine learning technologies, establish mapping relationships between inputs and outputs to achieve automatic learning and efficient prediction of inversion models. Compared with traditional physics-based inversion methods, data-driven approaches do not rely on analytical models or explicit solution processes, enabling the direct establishment of relationships between observational data and target parameters. Algorithms such as deep neural networks, convolutional networks, and variational autoencoders are widely applied in such inversion tasks. Their powerful feature extraction capabilities allow for effective modeling of complex geological response patterns without prior assumptions. Through end-to-end training, data-driven inversion significantly reduces dependence on traditional forward modeling modules while improving computational efficiency.

In structural construction, generalization capability, interpretability, and physical consistency are key indicators to consider. To address the strong “black-box” nature of traditional neural network structures, research has increasingly shifted toward physics-informed neural networks (PINNs), which embed partial differential constraints and boundary conditions into network training, ensuring that model outputs comply with fundamental physical laws. Additionally, hybrid inversion models integrate physical and data-driven models to form multi-level and multi-scale information fusion architectures, accommodating the nonlinearities and noise uncertainties inherent in complex geophysical environments. The representativeness of training datasets and the design of data augmentation methods also directly affect model accuracy and robustness, forming a crucial foundation for constructing high-reliability inversion frameworks.

3.3 Evolution Path and Future Trends of Dynamic Inversion Models

Traditional inversion models are mostly based on static data, making it difficult to address parameter variations and structural reconstructions during the temporal evolution of geological systems. Dynamic inversion models, by introducing the time dimension, treat geophysical data as time-varying signals, enabling continuous inversion and dynamic tracking of subsurface structures at different time points. Dynamic inversion not only enhances the interpretative capability regarding the evolution of subsurface media but also improves the adaptability of models to non-stationary systems. Such models are typically constructed based on time-series analysis, Kalman filtering, and particle filtering to establish time-update mechanisms, and they incorporate prior trajectory information to achieve collaborative parameter updates in the time domain [6].

Regarding future development trends, dynamic inversion models are evolving in three main directions: first, multi-scale models integrating time, space, and frequency dimensions, which improve sensitivity to dynamic processes through frequency-domain analysis and wavelet decomposition; second, online update models with enhanced data-stream input and real-time feedback capabilities, which enable continuous data acquisition and edge computing in Internet of Things (IoT) environments; third, self-evolving model structures, which adopt reinforcement learning and meta-learning strategies to autonomously adjust parameters and optimize solution paths, significantly improving response speed and generalization performance in complex geological scenarios. The modeling concept of dynamic inversion has already surpassed the limitations of static modeling paradigms and is gradually forming an intelligent inversion system with a closed-loop mechanism of “evolutionary cognition–predictive optimization–adaptive feedback.”

Conclusion

Based on the multiple requirements of geophysical modeling for accuracy, stability, and generalization capability, this study systematically reviews the structural composition, algorithmic logic, and model adaptation mechanisms of mathematical inversion theory, with a particular focus on the construction techniques of forward–inversion collaborative systems, high-dimensional regularization strategies, and multi-source fusion models. With the current trend of geosystem modeling shifting from static reconstruction to dynamic evolution, the intelligentization, multi-scaling, and temporalization of inversion structures have become key directions for future development. Multi-physical field coupling mechanisms enhance the comprehensive interpretability of solutions, physics-guided data-driven models expand the applicability of inversion algorithms, and dynamic inversion structures provide technical support for time-varying analysis of geological systems. Future research may further focus on

deep fusion mechanisms for multi-source heterogeneous data, interpretable learning frameworks constrained by physical laws, and construction pathways for inversion models with adaptive and evolutionary capabilities, thereby promoting the practical application and methodological innovation of inversion theory in complex geological systems.

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